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RANDOM DATA ANALYSIS

READY REFERENCE

The average properties of a random signal can be defined by several STATISTICAL parameters.

The signal may consist of two amplitude components—a mean (dc) value, plus a fluctuating (ac) value, i.e. $x(t) = ac + dc$.

(i) The MEAN value, μ_x, is the dc component, given by:

$$\mu_x = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt$$

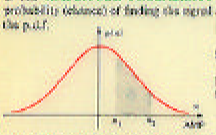
(ii) The MEAN SQUARE VALUE, Ψ_x^2, depends on both the ac and dc components:

$$\Psi_x^2 = ac^2 + dc^2 = \sigma_x^2 + \mu_x^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T x^2(t)dt$$

where σ_x = root mean square value of the ac component—sometimes called the STANDARD DEVIATION.

Probability

1. The AMPLITUDE PROBABILITY DENSITY FUNCTION, $p(x)$, gives the probability (chance) of finding the signal at a given amplitude x. $p(x)$ is often called the p.d.f.



Shaded area $= \int_{x_1}^{x_2} p(x)dx$ which is the probability of finding the signal x between the levels x_1 and x_2.

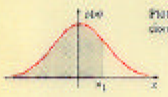
The p.d.f. is related to the two amplitude components:

(i) $\mu_x = \int_{-\infty}^{\infty} x p(x) dx$ (ii) $\Psi_x^2 = \int_{-\infty}^{\infty} x^2 p(x) dx$

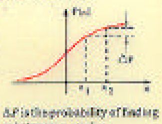
also $\int_{-\infty}^{\infty} p(x)dx = 1$.

2. The CUMULATIVE AMPLITUDE DISTRIBUTION FUNCTION, $P(x)$, gives the probability of finding the signal at or below a given amplitude x. $P(x)$ is often called the c.d.f.

$P(x_1) = \int_{-\infty}^{x_1} p(x)dx$ which is the shaded area shown below (left).



Plotting $P(x)$ as a function of x gives



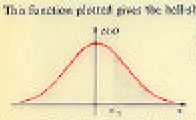
ΔP is the probability of finding x between x_1 and x_2.

3. The GAUSSIAN or NORMAL DISTRIBUTION is the most commonly encountered distribution for naturally occurring random signals, with a p.d.f. of the form:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

where σ is the root mean square value of x, if the mean of x is zero.

This function plotted gives the bell-shaped curve below.



The expression for $p(x)$ is frequently normalised with respect to the RMS value, σ, by making the substitution $u = x/\sigma$. The normalised p.d.f. is given by:

$$p(u) = \frac{1}{\sqrt{2\pi}} \cdot e^{-u^2/2}$$

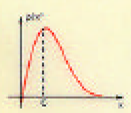
An ordinate table for positive values of u is given in Table 1, together with the shaded area, which is $[1-P(u)]$. The curve is symmetrical about the vertical axis. Table 2 gives $[1-P(u)]$ for highly improbable events.

u	p(u) ordinate	[1 − P(u)] shaded area
0	0·3989	5×10^{-1}
0·5	0·3521	$3{\cdot}085\times10^{-1}$
1·0	0·2420	$1{\cdot}587\times10^{-1}$
1·5	0·1295	$6{\cdot}68\times10^{-2}$
2·0	0·0540	$2{\cdot}28\times10^{-2}$
2·5	0·0175	$6{\cdot}21\times10^{-3}$
3·0	0·0044	$1{\cdot}35\times10^{-3}$
3·5	0·0009	$2{\cdot}33\times10^{-4}$

TABLE 1

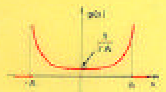
u	[1 − P(u)]
4	$2{\cdot}906\times10^{-5}$
5	$2{\cdot}90\times10^{-7}$
6	$8{\cdot}744\times10^{-10}$
7	$1{\cdot}134\times10^{-12}$
7·5	$2{\cdot}828\times10^{-14}$
12	$\simeq 10^{-33}$
15	$\simeq 10^{-50}$

TABLE 2

4. 

If narrowband Gaussian noise is passed through a rectifier, the envelope of the resulting signal has a RAYLEIGH DISTRIBUTION with a p.d.f. of the form:

$$p(x) = \frac{x}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}} \text{ for } x \geq 0$$

5. 

The p.d.f. of a SINE WAVE is given by:

$$p(x) = \frac{1}{\pi\sqrt{A^2 - x^2}} \quad \text{for } |x| \leq A$$

$$p(x) = 0 \quad \text{for } |x| > A$$

where A = peak amplitude.

Correlation

1. The AUTOCORRELATION FUNCTION, $R_x(\tau)$, is a measure of the similarity between a signal and a time-delayed form of itself, given by:

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)\cdot x(t-\tau)dt \text{ where } \tau \text{ is the time delay.}$$

The properties of this function are:

(i) $R_x(\tau)$ is a maximum positive value at $\tau = 0$, specifically $R_x(0) = \Psi_x^2$.

(ii) $R_x(\tau)$ tends to the square of the mean value as $\tau \to \infty$, if the signal is random.

(iii) $R_x(\tau) = R_x(-\tau)$, the autocorrelation function is symmetrical for positive and negative values of τ.

(iv) A periodic signal has a periodic autocorrelation function with the same period as the signal.

(v) If two uncorrelated signals are added together, the autocorrelation function of the resulting signal is equal to the sum of the separate autocorrelation functions, i.e. if $z = x + y$ then $R_z = R_x + R_y$.

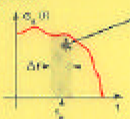
2. The CROSSCORRELATION FUNCTION, $R_{xy}(\tau)$, is a measure of the similarity between two signals as a function of a time shift, τ, between them, given by:

$$R_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t-\tau)\cdot y(t)dt$$

where one signal, $x(t)$, is delayed an amount τ with respect to the other signal, $y(t)$.

Spectra

1. The POWER SPECTRAL DENSITY FUNCTION is a measure of the power contained in a narrow frequency bandwidth Δf, centred on a frequency f_0. This can be expressed either for positive frequencies only (single-sided spectrum) or for positive and negative frequencies (double-sided spectrum). The latter method can never be measured in practice but frequently makes calculations simpler. The double-sided spectrum is symmetrical about the $f = 0$ axis. The frequency can either be expressed as ω or f (where $\omega = 2\pi f$). In the following expressions, f has been used. At any frequency f, the amplitude of the double-sided spectrum is one half that of the single-sided.



Power in this band $= G_x(f_0).\Delta f$

Total area under the curve $= \int_0^\infty G_x(f)df$

= Total power of signal $= \Psi_x^2$

The autocorrelation function and power spectral density function are a FOURIER TRANSFORM PAIR.

The single-sided Fourier transform pair equations are:

$$R_x(\tau) = \int_0^\infty G_x(f).\cos(2\pi f\tau)df; \quad G_x(f) = 4\int_0^\infty R_x(\tau)\cos(2\pi f\tau)d\tau$$

The double-sided Fourier transform pair equations are:

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\cos(2\pi f\tau)df; \quad S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\cos(2\pi f\tau)d\tau$$

Some examples of the single-sided pairs are given in Table 3.

Signal	$R_x(\tau)$	$G_x(f)$	$R_x(\tau)$	$G_x(f)$
Constant			c^2	$c^2.\delta(f)$
Sine Wave			$\frac{a^2}{2}\cos[2\pi f_0\tau]$	$\frac{a^2}{2}.\delta(f-f_0)$
White Noise			$\frac{a}{2}.\delta(\tau)$	$= a$ for $f \geq 0$ $= 0$ elsewhere
Lowpass White Noise			$\frac{aB.\sin[2\pi B\tau]}{[2\pi B\tau]}$	$= a$ for $0 \leq f \leq B$ $= 0$ elsewhere
Bandpass White Noise			$\frac{aB.\sin[\pi B\tau]}{[\pi B\tau]}\times[\cos(2\pi f_0\tau)]$	$= a$ within band $0 =$ elsewhere
Telegraph Wave (Random Binary Signal)			$\frac{-V^2}{\Delta T}.\tau + V^2$ for $\tau > 0$ $\frac{V^2}{\Delta T}.\tau - V^2$ for $\tau < 0$	$\frac{V^2\Delta T}{2}\left[\frac{\sin^2(\pi f\Delta T)}{(\pi f\Delta T)}\right]$

TABLE 3

Note—$\delta(x)$ is the Dirac delta function. $= 0$ for $x \neq 0$

$= \infty$ for $x = 0$

Total area under curve is unity.

2. The CROSS SPECTRAL DENSITY FUNCTION, $G_{xy}(f)$. As the power spectral density function and autocorrelation function are a Fourier transform pair, so are the cross-spectral density function and the cross-correlation function. $G_{xy}(f)$ is the single-sided spectrum (positive frequencies only).

The Fourier transform pair equations are:

$$G_{xy}(f) = 2\int_0^\infty R_{xy}(\tau)e^{-j2\pi f\tau}d\tau = C_{xy}(f) - jQ_{xy}(f)$$

where $C_{xy}(f)$ = Co-spectral Density Function

and $Q_{xy}(f)$ = Quadrature Density Function

$$R_{xy}(\tau) = \int_0^\infty [C_{xy}(f)\cos(2\pi f\tau) + Q_{xy}(f)\sin(2\pi f\tau)]df$$

Variance

A true AVERAGE requires summation over infinite time. This can never be done in practice, thus all statistical measurements are ESTIMATES. An example of an estimate of the mean value of a signal could be—

$$\hat{\mu}_x = \frac{1}{T}\int_0^T x(t)dt$$

where T is the total experiment time and the hat (∧) on the parameter indicates the estimated, rather than the actual, value. Since all practical measurements are estimates, errors arise called ESTIMATION ERRORS. The error in the measurement of a random signal is itself a random variable.

This consists of two components, a mean (dc) level called the BIAS ERROR and a fluctuating (ac) component called the VARIANCE. A common measure of the error introduced by measurement is the NORMALISED ERROR, defined:

$$\epsilon = \sqrt{\frac{\text{variance of parameter estimate}}{(\text{parameter})^2}}$$

The magnitude of the variance can be calculated for various measurement situations using Table 4. A quantity, $C_x(\tau)$, appears in the table, which is the autocorrelation function of the fluctuating component of the signal, called the AUTOCOVARIANCE FUNCTION, and is defined:

$$C_x(\tau) = R_x(\tau) - \mu_x^2$$

Φ	Variance of estimate of Φ. General expression	Variance of estimate of Φ for bandwidth limited white noise. Bandwidth = B
$\hat{\mu}_x$	$\frac{1}{T}\int_{-\infty}^{\infty} C_x(\tau)d\tau$	$\frac{\sigma_x^2}{2BT}$
$\hat{\Psi}_x^2$	$\frac{2}{T}\int_{-\infty}^{\infty}[C_x^2(\tau) + 2\mu_x^2 C_x(\tau)]d\tau$	$\frac{\sigma_x^4}{BT}$
$\hat{\sigma}_x^2$	$\frac{2}{T}\int_{-\infty}^{\infty} R_x^2(\tau)d\tau$	$\frac{\sigma_x^4}{BT}$
$\hat{p}(x)$	no simple expression	$\frac{p(x)}{2BTW}$ W = amplitude window
$\hat{R}_x(\tau)$	$\frac{1}{T}\int_{-\infty}^{\infty}[R_x^2(\tau) + R_x^2(0)]d\tau$	$\frac{1}{2BT}[R_x^2(0) + R_x^2(\tau)]$
$\hat{R}_{xy}(\tau)$	$\frac{1}{T}\int_{-\infty}^{\infty}[R_x(u).R_y(u) + R_{xy}(u+\tau)R_{xy}(u-\tau)]du$	$\frac{1}{2BT}[R_{xy}^2(\tau) + R_x(0)R_y(0)]$
$\hat{G}_x(f)$	$\frac{G_x^2(f)}{B_e T}$	$\frac{G_x^2(f)}{B_e T}$ B_e = filter bandwidth

TABLE 4

Bandwidth

In practical systems, filters and signals never have the ideal rectangular characteristics, so some convention has to be established for defining BANDWIDTH. Three definitions are in general use, described here with respect to a filter.

(i) 3 dB or HALF-POWER POINTS BANDWIDTH is the frequency interval between the upper and lower frequencies where the output of the filter attenuates the input signal by 3 dB, i.e., Power out = half of the Power in.

(ii) The NOISE BANDWIDTH is the bandwidth of a hypothetical rectangular filter which would pass the same power as the practical filter when the input to both is white noise.

(iii) The EQUIVALENT STATISTICAL BANDWIDTH. If a measurement using a hypothetical rectangular filter has the same mean square error (ϵ^2) as a measurement using the actual filter, when the input to both is white noise, the bandwidth of the hypothetical filter is the equivalent statistical bandwidth of the actual filter. (This is the value of B used in table 4.)

Further Reading

1. 'Measurement and Analysis of Random Data' by Julius S. Bendat and Allan G. Piersol (published by John Wiley and Sons, 1966).
2. 'Spectral Analysis and its Applications' by Gwilym M. Jenkins and Donald G. Watts (published by Holden Day, 1968).